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AN ATTACK-DEFENSE GAME WITH MATRIX STRATEGIES

N. D. Cohen

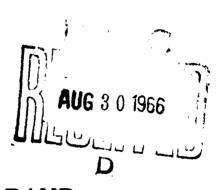*The* RAND *Corporation*

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PREFACE

Increasingly, the mathematical theory of games is finding interesting applications in the study of military conflict situations. This Memorandum examines an attack-defense game involving allocation of resources.

This application of the mathematical theories to a particular strategy game should be of interest both to mathematicians and to those directly involved in studies and analyses of military conflict situations.

Shortly after the release of RM-4274-PR, an error was discovered in the derivation of the results that required the issuance of this revision. The results quoted here are slightly less general than those given in the original version. This deficiency is overcome to a certain extent by the inclusion of a second payoff function and a more realistic example.

ABSTRACT

This paper presents the results and the method of analysis for an attack-defense game involving allocation of resources. Each player is assumed to have several different types of resources to be divided in an optimal fashion among a fixed set of targets. The payoff function of the game is convex.

The "No Soft-Spot" principle of M. Dresher,[*] and the concept of the generalized inverse of a matrix are used to determine optimal strategies for each player and the value of the game.

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[*]Dresher, M. Games of Strategy: Theory and Applications, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.

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AN ATTACK-DEFENSE GAME WITH MATRIX STRATEGIES[*]

N. D. Cohen

1. INTRODUCTION

Karlin [1] and Dresher [2] discussed an attack-defense game in which each of two players has a fixed amount of resources to be allocated among a set of targets. The attacker wishes to maximize, and the defender minimize, the amount of damage inflicted on the targets. These authors assume a convex payoff function and derive an optimal strategy for each player and the game value.

This paper generalizes these results. The resources of each player are divided into a fixed number of types, each type comprising a fixed percentage of the whole. Specifically, the attacker (Blue) has A resource units divided into s types, the mth type consisting of a_m units with

$$(1.1) \qquad \sum_{m=1}^{s} a_m = A .$$

Similarly, the defender (Red) has D units divided into r types, the jth type consisting of d_j units with

$$(1.2) \qquad \sum_{j=1}^{r} d_j = D .$$

The mth type of resource unit, if unopposed, can earn for the attacker a unit payoff ϵ_m, independent of the target. Further, each target $T_i (1 \leq i \leq n)$ has a unit value $\gamma_i > 0$. That is, each unopposed attacking unit of type m at the ith target, will earn for Blue a payoff $\gamma_i \epsilon_m$. Finally, the attacker is at least as strong as the defender ($A \geq D$), and the targets are ordered so that

$$(1.3) \qquad \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n .$$

Introducing distinct types of offensive and defensive units requires the defender to determine what percentage of his forces at each target will be expended on each type of offensive unit. To simplify the present analysis, we

will assume that this decision process is defined by a matrix $\Lambda = (\lambda_{mj})$, where λ_{mj} denotes that fraction of the allocated defensive units of type j to be used against an offensive unit of type m at any target. This definition implies that

$$(1.4) \qquad 0 \le \lambda_{mj} \le 1 \qquad (1 \le j \le r,\ 1 \le m \le s) ,$$

$$\text{and} \qquad \sum_{m=1}^{s} \lambda_{mj} = 1 .$$

The types of resource units available to the players will partially determine the assignment of values to the λ_{mj}'s. For example, torpedos cannot be used against planes, but neutralizing an aircraft carrier may require several torpedos (or many planes). Since the defender's optimal strategy and the game value depend strongly on the elements of Λ (see Sec. 5), a proper choice of these values may greatly decrease the attacker's payoff.

For the sake of definiteness, we will assume henceforth that $s \le r$. Then the column vectors of Λ are linearly independent. If they were not, then two or more types of defensive units could be combined into a single type without

loss of generality. Therefore, the rank of Λ equals s.

If s > r, we must work with the transpose of Λ. In this

case, the row vectors of Λ' are linearly independent and

the rank of Λ' equals r.

2. PAYOFF FUNCTION

The payoff function of the game will in general re-
flect the aims of the attacker. If the only consideration
is inflicting damage on the targets, then the attacker will
only be concerned with achieving a numerical superiority
for each of his types of resources at each target. Thus,
if at the ith target, the attacker's forces of resource
type m are not numerically superior to the defending forces
opposing them, then the payoff to the attacker corresponding
to that resource type is zero.

On the other hand, the attacker may be equally concerned
with preventing the defender from achieving an offensive role
at some future time. In this case, the attacker must assume
to have received no advantage (zero payoff) unless at any
target his total forces are numerically superior to the de-
fender's total forces. The following discussion refers to
these two different philosophies as Case I and Case II.

The foregoing remarks imply several simplifying as-
sumptions common to most game theoretic solutions of military
conflict situations. First, the game consists of a single
move during which the players act simultaneously. Second,
the targets are independent of one another (i.e., an attack

on one of them has no effect on any other), opposing forces
of equal strength neutralize each other, and finally, a
commitment of forces once made by a player cannot be changed.

With the above restrictions in mind and to retain the
spirit of [1] and [2], we will assume the following payoff
functions (for Blue).

Case I:

$$(2.1) \quad M(x,y) = \sum_{i=1}^{n} \gamma_i \sum_{m=1}^{s} \max \left\{ 0 \ , \ \epsilon_m \left(x_{im} - \sum_{j=1}^{r} \lambda_{mj} y_{ij} \right) \right\} \ .$$

Case II:

$$(2.2) \quad M(x,y) = \sum_{i=1}^{n} \gamma_i \max \left\{ 0 \ , \ \sum_{m=1}^{s} \epsilon_m \left(x_{im} - \sum_{j=1}^{r} \lambda_{mj} y_{ij} \right) \right\}$$

where x_{im} and y_{ij} denote, respectively, the attacking forces
of type m and the defending forces of type j assigned by
each player to the ith target. Clearly, each payoff func-
tion is convex in y for each x and convex in x for each y.

3. GAMES WITH CONVEX PAYOFF FUNCTIONS IN E^n

Let M(x,y) be a payoff function defined for all y in some convex compact set Y in E^n, and for all x in some compact set X. Assume that M(x,y) is jointly continuous in x and y, and convex in y for each x∈X. Let U and W be the mixed strategy spaces of the two players, i.e., all distribution functions over X and Y, respectively. Games of this type have solutions characterized by the following theorem of Bohnenblust, Karlin, and Shapley [4].

THEOREM

Let M(x,y) be as described above. Then the function

$$\varphi(y) = \sup_{x} M(x,y)$$

achieves its minimum at some point $y_o \in Y$. Further, there exist numbers $\{\epsilon_i^o\}$, $\{x_i^o\}$ such that

$$\sum_{i=1}^{n+1} \epsilon_i^o M(x_i^o,y) \geq \varphi(y_o)$$

for all y∈Y, where $\epsilon_i^o \geq 0$ and $\sum_{i=1}^{n+1} \epsilon_i^o = 1$.

Thus, if Blue uses the finite mixed strategy

$$u^*(x) = \sum_{i=1}^{n+1} \epsilon_i^o \, u_{x_i^o}(x) \, ,$$

his payoff will be at least $\varphi(y_o)$. But if Red uses the pure strategy y_o, then for any $x \in X$ the payoff to Blue is

$$M(x,y) \leq \sup_x M(x,y_o) = \varphi(y_o) \, .$$

Therefore, $\varphi(y_o)$ is the value of the game, and $\{u^*, y_o\}$ are optimal strategies for Blue and Red, respectively.

If X is a simplex with vertices a_i $(1 \leq i \leq n)$, then every $x \in X$ has a representation of the form

$$\sum_{i=1}^{n} \mu_i a_i \, , \quad \mu_i \geq 0 \, , \quad \sum_{i=1}^{n} \mu_i = 1 \, .$$

Thus, if $M(x,y)$ is also convex in x for each $y \in Y$, then

$$M(x,y) = M\left(\sum_{i=1}^{n} \mu_i a_i, y \right) \leq \sum_{i=1}^{n} \mu_i M(a_i, y) \, ,$$

so that

$$\sup_{x} M(x,y) = \sup_{i} M(a_i,y).$$

It is therefore sufficient for Blue to randomize over the vertices of the simplex X.

4. THE "NO SOFT-SPOT" PRINCIPLE

The variables of the payoff functions (2.1) and (2.2) range over simplices whose vertices may be represented by matrices

$$\mathcal{A}_i = (0,0,\ldots,a,\ldots,0)$$

$$(1 \leq i \leq n)$$

$$\mathcal{D}_i = (0,0,\ldots,d,\ldots,0)$$

respectively. The vectors $a = (a_1, a_2, \ldots, a_s)'$ and $d = (d_1, d_2, \ldots, d_r)'$ occupy the ith column in each matrix, and the remaining elements are zero. These vertices represent the pure strategies for Blue and Red, respectively. Applying the theorem to the game described in Secs. 1 and 2, we have the following:

1) Blue has an optimal mixed strategy which consists of allocating his entire force to a single target chosen by means of a probability-distribution function;

2) Red has the optimal pure strategy (the y_o that minimizes $\sup_x M(x,y)$) of allocating each type of defensive unit over the n targets;

3) The value of the game is $\min_y \sup_x M(x,y)$.

While the theorem of Sec. 3 completely characterizes the solution of the game, it does not provide any practical means for determining either the optimal strategies or the value of the game. We achieve this by using Dresher's "No Soft-Spot" principle [2,3], which states that an optimal strategy for Red is to defend only those targets which, under a concentrated attack, would yield Blue the value of the game. Conversely, Blue should attack only those targets which Red chooses to defend.

5. THE SOLUTION

The technique to be used is applicable to the payoff function of both Case I and Case II. However, since the latter is mathematically more interesting, we restrict the derivation of results to Case II. Section 6 quotes the corresponding results for Case I.

Recall that the payoff to Blue is

$$(5.1) \qquad M(x,y) = \sum_{i=1}^{n} \gamma_i \ \max \left\{ 0 \ , \ \sum_{m=1}^{s} \epsilon_m \left(x_{im} - \sum_{j=1}^{r} \lambda_{mj} y_{ij} \right) \right\} .$$

At the ith target, Blue and Red each choose a column vector whose elements represent the amounts of each type of resource unit allocated to that target: $X_i = (x_{im})$, $Y_i = (y_{ij})$, $(1 \le m \le s, \ 1 \le j \le r)$, respectively. In addition, define column vectors $G_i = (\gamma_i \epsilon_m)$, $V_i = (v_m / \gamma_i \epsilon_m)$, where v_m is that portion of the game value v contributed by the attacking units of type m and

$$\sum_{m=1}^{s} v_m = v .$$

Using this notation, the payoff function may be written
as

$$(5.2) \qquad M(x,y) = \sum_{i=1}^{n} \max \left\{ 0 \,,\, G_i' \left(X_i - A Y_i \right) \right\} \quad .$$

If Blue commits his entire force to the ith target,
he must receive (by the "No Soft-Spot" principle) exactly
v. Thus, replacing X_i by the vector $Q = (a_m)$ yields

$$(5.3) \qquad G_i' (Q - A Y_i - V_i) = 0 \; .$$

The left side of Eq. (5.3) represents the inner product of
two vectors in an s-dimensional vector space. Hence, it
will be satisfied by any vector orthogonal to G_i. Since,
in particular, any vector is orthogonal to the zero vector,

$$(5.4) \qquad A Y_i = Q - V_i \; .$$

In order to solve Eq. (5.4) for Y_i, we introduce the
concept of the generalized inverse (g.i.) for a matrix,
due to Moore [5] and Penrose [6,7]. The g.i. exists for
any (possibly rectangular) matrix with real or complex

elements. Moreover, it is unique. The g.i. of a matrix A (written $A^\dagger$) is that X which satisfies the four matrix equations

$$(5.5) \qquad AXA = A \ ,$$

$$(5.6) \qquad XAX = X \ ,$$

$$(5.7) \qquad (AX)^* = AX \ ,$$

$$(5.8) \qquad (XA)^* = XA \ ,$$

where A^* denotes the conjugate transpose of A. The matrix A need not be square, and may even consist entirely of zero elements. If A is non-singular, then $A^\dagger = A^{-1}$. In particular, if α is a scalar, then $\alpha^\dagger$ means α^{-1} if $\alpha \neq 0$, and 0 if $\alpha = 0$. If AA^* is non-singular, then Eqs. (5.5) and (5.8) yield

$$(AXA)^* = (XA)^* A^* = XAA^* = A^*$$

and, therefore,

$$(5.9) \qquad X = A^\dagger = A^* (AA^*)^{-1} \ .$$

Consider the matrix equation $BZ = C$. A necessary and sufficient condition for this equation to have a solution is that $BB^\dagger C = C$ [6,9], in which case the general solution is given by

$$(5.10) \qquad Z = B^\dagger C + F(I - B^\dagger B) \ ,$$

where F is an arbitrary matrix.

Returning to Eq. (5.4), recall that we assumed $\Lambda\Lambda^*$ to be non-singular. This implies, from Eq. (5.9), that $\Lambda\Lambda^\dagger = I$ (identity matrix), which is clearly a sufficient condition for Eq. (5.4) to have a solution of the form of Eq. (5.10).

Since the matrix F in Eq. (5.10) is arbitrary, it may be taken equal to the zero matrix Thus, a particular solution of Eq. (5.4) is given by

$$(5.11) \qquad Y_i = \Lambda^\dagger (Q - V_i) \ .$$

Let $\Lambda^\dagger = (\beta_{jm})$. Then Eq. (5.11) and the definition of Y_i yields

$$(5.12) \qquad y_{ij} = \sum_{m=1}^{s} \left(a_m - \frac{v_m}{\gamma_i \epsilon_m} \right) \beta_{jm} \qquad (1 \le j \le r) \; .$$

By the "No Soft-Spot" principle, Red should allocate all the defensive units of type j among the $t(j) \le n$ most valuable targets. Summing Eqs. (5.12) over these targets and rearranging terms yields

$$(5.13) \qquad \sum_{m=1}^{s} \frac{v_m}{\epsilon_m} \beta_{jm} = \left(t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right) \Big/ L_{t(j)} \qquad (1 \le j \le r) \; ,$$

where

$$L_{t(j)} = \sum_{i=1}^{t(j)} 1/\gamma_i \; .$$

If we define the column vectors $V = (v_m/\epsilon_m)$, $R = (d_j/t(j))$, and the matrix $S = \text{diag} \left(t(j)/L_{t(j)} \right)$ $(1 \le m \le s, \; 1 \le j \le r)$, then Eqs. (5.13) are equivalent to the matrix equation

$$(5.14) \qquad \Lambda^* V = S(\Lambda^* Q - R) \; .$$

Premultiplying both sides of Eq. (5.14) by Λ, and recalling that $\Lambda\Lambda^* = I$, gives

(5.15) $V = \Lambda S(\Lambda^\dagger Q - R)$

or

(5.16) $v_\iota = \epsilon_\iota \sum_{j=1}^{r} \lambda_{\iota j} \left(t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right) \Big/ L_{t(j)} \quad (1 \leq \iota \leq s)$.

We wish to choose the numbers $t(j) \leq n$ so as to maximize v. Clearly, it is sufficient to individually maximize each of the terms

(5.17) $\iota(t(j)) = \sum_{\iota=1}^{s} \epsilon_\iota \lambda_{\iota j} \left(t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right) \Big/ L_{t(j)}$.

In fact, we will show that the game value is a concave function of $t(j)$. To do this, we must show that

$\iota(t(j) + 2) - 2\iota(t(j) + 1) + \iota(t(j)) \leq 0$.

Note that condition (1.3) implies that

$L_{t(j)+2} \geq L_{t(j)+1} \geq L_{t(j)}$.

Letting

$$(5.18) \qquad \varphi(t(j)) = \sum_{\ell=1}^{s} \epsilon_\ell \left(t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right) \Big/ L_{t(j)}$$

and recalling that $0 \le \lambda_{\ell j} \le 1$, then

$$(5.19) \qquad \psi(t(j) + 2) - 2\psi(t(j) + 1) + \psi(t(j)) \le \varphi(t(j) + 2)$$

$$- 2\varphi(t(j) + 1) + \varphi(t(j)) \le \sum_{\ell=1}^{s} \epsilon_\ell / L_{t(j)}$$

$$\left[(t(j) + 2) \sum_{m=1}^{s} a_m \beta_{jm} - d_j - 2(t(j) + 1) \sum_{m=1}^{s} a_m \beta_{jm} + 2d_j \right.$$

$$\left. + t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right] = 0$$

which is the required condition. Thus, Red will allocate his resources of type j among the $t(j) \le n$ most valuable targets such that $\psi(t(j))$ is a maximum.

By the "No Soft-Spot" principle, Blue will attack, with his entire force, only defended targets. At all such targets, his expected unit payoff will be the same. Letting

P_i be the probability that Blue will attack the ith target,

and letting

$$t = \max_j t(j)$$

we have

(5.20) $P_i \gamma_i = C$ (constant) ,

and

(5.21) $\displaystyle\sum_{i=1}^{t} P_i = 1$.

A few steps, then, yield

(5.22) $P_i = 1/\gamma_i L_t$ $(1 \le i \le t)$,

          $P_i = 0$ $(i > t)$.

THEOREM

(i) The game value is non-negative.

(ii) The pure strategy for Red given by Eq. (5.12)
and the mixed strategy for Blue given by Eq.
(5.22) are both optimal.

<u>Proof</u>:

To prove (i), let

$$L = \max_{j} L_{t(j)} \quad, \quad \epsilon = \min_{\ell} \epsilon_{\ell}$$

Then

$$v \geq (\epsilon/L) \left\{ \sum_{\ell=1}^{s} \sum_{j=1}^{r} \lambda_{\ell j} \left(t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right) \right\}$$

$$\geq (\epsilon/L) \left\{ \sum_{\ell=1}^{s} \sum_{j=1}^{r} \sum_{m=1}^{s} a_m \lambda_{\ell j} \beta_{jm} - \sum_{\ell=1}^{s} \sum_{j=1}^{r} \lambda_{\ell j} d_j \right\}$$

$$= (\epsilon/L) \{A - D\} \geq 0 .$$

In order to prove (ii), we must show that Red's ex-
pectation is at most v and Blue's expectation at
least v. Integrating the payoff function with re-
spect to Red's optimal strategy yields

$$E_R = \gamma_i \sum_{\ell=1}^{s} \epsilon_\ell \left\{ x_{i\ell} - \sum_{j=1}^{r} \lambda_{\ell j} \left[\sum_{m=1}^{s} \left(a_m - \frac{v_m}{\gamma_i \epsilon_m} \right) \beta_{jm} \right] \right\}$$

$$\leq \gamma_i \left\{ \sum_{\ell=1}^{s} \epsilon_\ell \left[a_\ell - \sum_{j=1}^{r} \sum_{m=1}^{s} a_m \lambda_{\ell j} \beta_{jm} \right. \right.$$

$$\left. \left. + \sum_{j=1}^{r} \sum_{m=1}^{s} \frac{v_m}{\gamma_i \epsilon_m} \lambda_{\ell j} \beta_{jm} \right] \right\}$$

$$= \gamma_i \left\{ \sum_{\ell=1}^{s} \epsilon_\ell a_\ell - \sum_{\ell=1}^{s} \epsilon_\ell a_\ell + \sum_{\ell=1}^{s} \frac{v_\ell}{\gamma_i} \right\}$$

$$= v .$$

Integrating the payoff function with respect to Blue's optimal strategy yields

$$E_B = \sum_{i=1}^{t} \gamma_i/\gamma_i L_t \sum_{\ell=1}^{s} \epsilon_\ell \left[a_\ell - \sum_{j=1}^{r} \lambda_{\ell j} y_{ij} \right]$$

$$= (t/L_t) \sum_{\ell=1}^{s} \epsilon_\ell \left[\sum_{j=1}^{r} \sum_{m=1}^{s} a_m \lambda_{\ell j} \beta_{jm} \right]$$

$$- \sum_{i=1}^{t} (1/L_t) \sum_{\ell=1}^{s} \epsilon_\ell \sum_{j=1}^{r} \lambda_{\ell j} y_{ij}$$

$$\geq \sum_{\ell=1}^{s} \epsilon_\ell \sum_{j=1}^{r} \left(t(j) \sum_{m=1}^{s} a_m \lambda_{\ell j} \beta_{jm} \right) \Big/ L_{t(j)}$$

$$- \sum_{\ell=1}^{s} \epsilon_\ell \sum_{j=1}^{r} \lambda_{\ell j} d_j \Big/ L_{t(j)}$$

$$= \sum_{\ell=1}^{s} \epsilon_\ell \sum_{j=1}^{r} \lambda_{\ell j} \left(t(j) \sum_{m=1}^{s} a_m \beta_{jm} - d_j \right) \Big/ L_{t(j)}$$

$$= \nu \; . \; \blacksquare$$

6. SOLUTION FOR CASE I

The results for Case I may be derived in a manner analogous to Case II, by treating each of Blue's resource types individually. The optimal strategies are

$$y_{ij} = \sum_{m=1}^{s} \left(\lambda_{mj}/\mu_m \right) \left(a_m - (v_m/\gamma_i \epsilon_m) \right)$$

$$(1 \le j \le r, \ 1 \le i \le t(j)) \ ,$$

where

$$\mu_m = \sum_{k=1}^{r} \lambda_{mk}^2 \ ,$$

and

$$P_i = 1/\gamma_i L_t \qquad\qquad (1 \le i \le t) \ ,$$

$$P_i = 0 \qquad\qquad (i > t) \ .$$

The game value is

$$v_\ell = \epsilon_\ell \mu_\ell \sum_{j=1}^{r} \beta_{\ell j} \left\{ t(j) \sum_{m=1}^{s} \left(a_m \lambda_{mj}/\mu_m \right) - d_j \right\} \Big/ L_{t(j)}$$

where $(\beta_{\ell j})$ are the elements of $\Lambda'^{\dagger}$ and

$$\sum_{\ell=1}^{s} v_{\ell} = v .$$

7. A NUMERICAL EXAMPLE

The Blue force is half planes (where $\epsilon_1 = 2$) and
half surface ships (where $\epsilon_2 = 1$). He is opposed by an
equal force of one-third planes, one-third submarines
carrying torpedos, and one-third surface ships. Red is
defending one large aircraft carrier ($\gamma_1 = 1$), two smaller
carriers ($\gamma_2 = \gamma_3 = 1/2$), and two tankers ($\gamma_4 = \gamma_5 = 1/4$).
Red decides that, at any target, his planes will be dis-
tributed equally between Blue's planes and ships, three-
fourths of his surface ships will oppose planes and one-
fourth surface ships. Clearly, torpedos can be used only
against surface ships. From this verbal description of
the game,

$$a_1 = a_2 = F/2 ,$$

$$d_1 = d_2 = d_3 = F/3 ,$$

$$L_1 = 1, \ L_2 = 3, \ L_3 = 5, \ L_4 = 9, \ L_5 = 13,$$

$$\Lambda = \begin{pmatrix} 1/2 & 0 & 3/4 \\ 1/2 & 1 & 1/4 \end{pmatrix}$$

Equation (5.9) gives

$$\Lambda^{\dagger} = \begin{pmatrix} 1/2 & 3/14 \\ -1/2 & 13/14 \\ 1 & -1/7 \end{pmatrix}.$$

Therefore,

$$\psi(t(1)) = (15t(1) - 14) \ F/28L_{t(1)} \ ,$$

$$\psi(t(2)) = (9t(2) - 14) \ I \cdot 42L_{t(2)} \ ,$$

$$\psi(t(3)) = (9t(3) - 7) \ F/12L_{t(3)} \ .$$

Evaluating these functions for $t(j) = 1, 2, 3, 4, 5$, we find that $t(1) = t(2) = t(3) = 3$. Thus Red will defend only the carriers. Substituting in Eq. (5.16), we get

$$v_1 = 13F/30, \ v_2 = 11F/60, \ v = 37F/60 \ .$$

Substitution in Eq. (5.12) yields for Red the optimal strategy:

$$y_{11} = 22F/105 \qquad y_{21} = 13F/210 \qquad y_{31} = 13F/210,$$

$$y_{12} = 16F/105 \qquad y_{22} = 19F/210 \qquad y_{32} = 19F/210,$$

$$y_{13} = 5F/21 \qquad y_{23} = F/21 \qquad y_{33} = F/21,$$

$$y_{ij} = 0, \quad i = 4,5$$

Finally, Eq. (5.22) shows that Blue should attack the large carrier with probability 1/5, and each of the two smaller carriers with probability 2/5.

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10. ABSTRACT	11. KEY WORDS
This Memorandum presents the results and the method of analysis for an attack-defense game involving allocation of resources. Each of the two players is assumed to have several different types of resources to be divided in an optimal fashion among a fixed set of targets. The payoff function of the game is convex. The "no soft-spot" principle of M. Dresher and the concept of the generalized inverse of a matrix are used to determine optimal strategies for each player and the value of the game.	Gaming Mathematics Matrices Strategy